

Perfect simulation for unilateral fields

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Abstract

In this paper we consider two-point unilateral Markov fields on a two-dimensional lattice as considered by Pickard [15], Galbraith and Walley [4, 5]. We show that, under various ergodicity conditions, they can be perfectly simulated in the stationary state on any finite window. The techniques which are used connect perfect simulation with oriented percolation through suitable coupling constructions.

Mathematics Subject Classification 2000: Primary 60K35, 65C05;

Secondary 60G60, 60G10.

Key words and phrases: Unilateral fields, Perfect simulation, Oriented percolation.

1 Introduction

In this paper we consider a particular class of random fields on the two-dimensional lattice \mathbb{N}^{+2} and their stationary extensions to \mathbb{Z}^2 , the so-called *unilateral* random fields. These fields appeared in the literature as models for crystal growth [17]; later they have been proposed also for image analysis applications [6]. Unilateral random fields can be defined in general w.r.t. an oriented acyclic graph structure (as in [10], where they are called Bayesian networks): for the sake of simplicity we will consider the so-called two-point models on \mathbb{Z}^2 , which corresponds to the nearest neighbor structure where oriented edges connect (i, j) with $(i + 1, j)$ and $(i, j + 1)$. However the reader will recognize that our basic idea works for more general translation invariant graphs.

A unilateral two-point field model is constructed through a transition kernel. We assume that the state space E is Borel, i.e. a Borel subset of a Polish space endowed with its Borel σ -algebra. A (unilateral two-point) transition kernel K on E is a mapping $K : E^2 \rightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the space of probability measures on E , with the property that for any measurable set $B \subset E$ the function $K(B|x, y) = K(x, y)(B)$ is measurable in the pair $(x, y) \in E^2$. An E -valued unilateral field $\mathbf{X}^{\mathbf{x}} = \{X_{i,j}^{\mathbf{x}} \in E, (i, j) \in \mathbb{N}^{+2}\}$ can be constructed from K for any choice of the boundary conditions (b.c.'s) $\mathbf{x} = \{x_{i,0}, x_{0,i}, i \in \mathbb{N}^+\}$. The construction is accomplished by specifying consistently the law of $\mathbf{X}_{\Lambda_{m,n}}^{\mathbf{x}}$ in any finite box

$$\Lambda_{m,n} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \quad (1)$$

as being

$$P(\mathbf{X}_{\Lambda_{m,n}}^{\mathbf{x}} \in B) = \int_B \prod_{i=1}^m \prod_{j=1}^n K(dx_{i,j} | x_{i-1,j}, x_{i,j-1}), \quad (2)$$

where B is any measurable set in $E^{m \times n}$. A natural way to simulate this field is to produce the variables $X_{i,j}^{\mathbf{x}}$ in any sequential (total) order such that each site (i, j) comes after its *parents* $(i-1, j)$ and $(i, j-1)$. A total order with this property will be called *increasing* in the sequel. The reader will immediately notice the similarity with discrete time Markov chains.

Next we randomize the b.c.'s, say with law μ . We define the law of the unilateral field \mathbf{X}^μ as

$$P(\mathbf{X}_{\Lambda_{m,n}}^\mu \in A) = \int P(\mathbf{X}_{\Lambda_{m,n}}^{\mathbf{x}} \in A) \mu(d\mathbf{x}). \quad (3)$$

We say that the b.c.'s are Markovian when the *horizontal boundary* $\mathbf{X}_{\cdot,0} = \{X_{i,0}, i = 1, 2, \dots\}$ and the *vertical boundary* $\mathbf{X}_{0,\cdot} = \{X_{0,i}, i = 1, 2, \dots\}$ are Markov chains which are conditionally independent given a common starting value $X_{0,0}$ (which can be considered as the value of the field at the origin). For Markovian b.c.'s the sequential simulation of the field \mathbf{X}^μ is easy, using any increasing order of the sites of \mathbb{N}^2 (including boundary sites).

Pickard [15] studied the problem of determining laws μ such that the corresponding field \mathbf{X}^μ is stationary, i.e. invariant under translations in \mathbb{N}^2 . Stationary unilateral fields can be extended to \mathbb{Z}^2 by Kolmogorov's theorem. When E is finite one can prove that stationary unilateral fields can always be constructed. Using again Kolmogorov's theorem, it suffices to take compatible convergent subsequences of the sequence of averages

$$\mu_{\Lambda_{m,n}}^L(\cdot) = \frac{1}{L^2} \sum_{t \in \Lambda_{L,L}} P(\mathbf{X}_{\Lambda_{m,n}+t}^{\mathbf{x}} \in \cdot), \quad L = 1, 2, \dots \quad (4)$$

in any fixed finite box $\Lambda_{m,n}$, and use standard compactness arguments.

For Markovian b.c.'s, Pickard established sufficient conditions for stationarity of a unilateral field, which become also necessary in the binary case. Pickard's conditions are formulated in terms of the joint distribution of the 4-tuple of random variables $(X_{0,0}^\mu, X_{1,0}^\mu, X_{0,1}^\mu, X_{1,1}^\mu)$ (see [15]), hence they involve K and μ . On the other hand, Galbraith and Walley have proved that in the binary case any positive two-point unilateral kernel K has a unique Markovian boundary law $\mu^K = \mu$ under which the columns $(X_{0,0}^\mu, X_{1,0}^\mu)$ and $(X_{0,1}^\mu, X_{1,1}^\mu)$ have the same law, and also the rows $(X_{0,0}^\mu, X_{0,1}^\mu)$ and $(X_{1,0}^\mu, X_{1,1}^\mu)$ have the same law [4]. Since these two conditions are necessary for a stationary field, they could also reformulate Pickard's conditions directly in terms of K .

However not all stationary unilateral fields have Markovian boundary laws, hence they are not necessarily easy to simulate sequentially. The goal of this paper is to construct simulation algorithms for the class of stationary unilateral fields which is introduced next.

We say that a unilateral field $\mathbf{X}^{\mathbf{x}}$ with kernel K is *ergodic* if there exists a stationary field $\mathbf{X}' = \{X'_{i,j} \in E, i, j = 0, 1, 2, \dots\}$ such that for every b.c.'s \mathbf{x} and every pair $(m, n) \in$

\mathbb{N}^2

$$\lim_{(h,l) \rightarrow \infty} \|\mathcal{L}(\mathbf{X}_{(h,l)+\Lambda_{m,n}}^{\mathbf{x}}) - \mathcal{L}(\mathbf{X}'_{\Lambda_{m,n}})\| = 0 \quad (5)$$

where we use $\mathcal{L}(\mathbf{Y})$ to denote the law of a random vector \mathbf{Y} , and $\|\cdot\|$ is the total variation. If the limit in (5) is uniform in the boundary conditions \mathbf{x} we say that the kernel is *uniformly ergodic*. In the binary case Galbraith and Walley [4] gave a quite involved sufficient condition for the uniform ergodicity of K .

We call the field \mathbf{X}' appearing in (5) the *equilibrium field* of the kernel K . It is easy to see that \mathbf{X}' is a stationary unilateral field with kernel K , and if it exists it is necessarily unique. Since its boundary law μ' could be very hard to compute (even in the binary case), it is not easy to simulate \mathbf{X}'_{Λ} even on a small box Λ . In fact, by (5) we know that we can only *approach* $\mathcal{L}(\mathbf{X}'_{\Lambda})$ by shifting the box *sufficiently far away* from the boundary. The problem is analogous to that of sampling a stationary Markov chain on a finite window when the stationary distribution is not available (as it happens in MCMC simulations); but in this case, in order to simulate a box, we need to determine the whole joint equilibrium law of its boundary.

Under a rather strong minorization condition on K (Assumption 2.1) for a general Borel state space E , we show in the next two sections how to implement a simulation algorithm which produces a sample on any finite box, exactly distributed as the equilibrium field. As a byproduct, we establish uniform ergodicity of K . For one-dimensional discrete-time Markov processes the first algorithm of this type was the CFTP algorithm of Propp and Wilson [16], see also [8]. This work stimulated a wide interest toward what is now called perfect simulation, see [12]. Our algorithm is based on an idea introduced by Murdoch and Green [13] for discrete-time Markov processes. It consists in coupling the whole family of unilateral fields, with all possible boundary conditions, by using an underlying auxiliary Bernoulli field. We show that the dependence from boundary values is propagated only along increasing open paths in such a Bernoulli field. In this way results from oriented percolation for two-dimensional Bernoulli fields can be used in order to ensure that the propagation stops with probability 1 when the boundary moves far away from the region to be simulated. In order to simulate a square box of side L , the algorithm requires, in addition to the L^2 variables of the box, an average of $O(L)$ additional random variables of the field. For other graphical constructions used in the study of ergodicity see e.g. [2].

In Section 4, in order to relax Assumption 2.1, we study a more general class of algorithms working on blocks of adjacent sites lying on selected diagonals. These algorithms work under the more general Assumption 4.1, which is however not equally easy to check. In Section 5, using this block algorithm, we present some examples of kernels for which Assumption 2.1 fails but nevertheless a perfect simulation algorithm can still be constructed.

2 Coupling of unilateral fields

In this section we introduce the auxiliary i.i.d. fields which allow to couple, i.e. to represent on the same probability space, the family of unilateral fields $\mathbf{X}^{\mathbf{x}}$ defined in (2), for all possible boundary conditions $\mathbf{x} \in E^{\mathbb{N}^+} \times E^{\mathbb{N}^+}$. Moreover we investigate when the values of these fields indicate that the dependence from the boundary conditions of the field in some fixed box is lost.

In the next two sections we suppose that the kernel K satisfies the following *minorization condition*.

Assumption 2.1. *There exists a probability measure ϕ on E and a positive constant $\delta \geq \delta_0$ such that*

$$K(A|y_1, y_2) \geq \delta \phi(A) \quad (6)$$

for every measurable set A and any pair $y_1, y_2 \in E$. δ_0 is a positive constant which will be specified in the next section.

If $\delta = 1$ the field is trivially i.i.d. so we exclude this possibility from now on.

Remark 2.2. *If E is finite or countable, then provided*

$$\tau(z) = \inf\{K(\{z\}|y_1, y_2) : y_1, y_2 \in E\} \quad (7)$$

is not identically zero, the minorization condition (6) is satisfied with

$$\delta = \sum_{z \in E} \tau(z) > 0, \quad (8)$$

and

$$\phi(\{z\}) = \frac{\tau(z)}{\delta}. \quad (9)$$

Under Assumption 2.1 it is immediately checked that

$$H(\cdot|y_1, y_2) = \frac{K(\cdot|y_1, y_2) - \delta \phi(\cdot)}{1 - \delta}$$

is a kernel on E . Since E is Borel we can always define a function $f : (0, 1) \times E^2$ which is separately measurable in each of its two arguments (see [9]) with the property that when U is uniformly distributed in the interval $(0, 1)$, then $f(U; y_1, y_2)$ has the law $H(\cdot|y_1, y_2)$ for any pair $(y_1, y_2) \in E^2$. Next we can prove the following

Lemma 2.3. *Let $\{Z, V, U\}$ be mutually independent random variables with laws*

$$P(Z = 0) = 1 - P(Z = 1) = \delta, \quad V \sim \phi, \quad U \text{ uniform on } (0, 1), \quad (10)$$

and define

$$g(z, v, u; y_1, y_2) = (1 - z)v + zf(u; y_1, y_2). \quad (11)$$

Then for any $y_1, y_2 \in E$

$$g(Z, V, U; y_1, y_2) \sim K(\cdot|y_1, y_2). \quad (12)$$

Proof. For any $y_1, y_2 \in E$ we can write the kernel as a mixture

$$K(\cdot|y_1, y_2) = \delta\phi(\cdot) + (1 - \delta)H(\cdot|y_1, y_2) \quad (13)$$

which is seen to be induced by the application of (11) to $\{Z, V, U\}$. \square

We say that the family of functions $\{g(\cdot|y_1, y_2), (y_1, y_2) \in E^2\}$ realizes a *coupling* of the family of laws $\{K(\cdot|y_1, y_2), (y_1, y_2) \in E^2\}$ on the probability space where Z, V , and U are defined. Notice that when $Z = 0$ all the random variables $g(Z, V, U; y_1, y_2)$ take the same value V , irrespectively of $(y_1, y_2) \in E^2$. In this case we say that coupling occurs. When E is finite or countable, with δ and ϕ defined as in Remark 2.2, the above coupling is *maximal*, since the probability of coupling can never exceed δ . In fact, if $\{X^{y_1, y_2}, (y_1, y_2) \in E^2\}$ is any other coupling of $\{K(\cdot|y_1, y_2), (y_1, y_2) \in E^2\}$, then

$$P(\exists x \in E : X^{y_1, y_2} = x, \forall (y_1, y_2) \in E^2) \leq \sum_x \inf_{y_1, y_2} P(X^{y_1, y_2} = x) = \delta. \quad (14)$$

Now let

$$\{\mathbf{Z}, \mathbf{U}, \mathbf{V}\} = \{Z_{i,j}, U_{i,j}, V_{i,j}\}_{(i,j) \in \mathbb{Z}^2}$$

be mutually independent i.i.d. fields with $(Z_{i,j}, U_{i,j}, V_{i,j}) \sim (Z, U, V)$ distributed as in (10).

From the previous lemma we see that the probability space supporting $\{\mathbf{Z}, \mathbf{U}, \mathbf{V}\}$ allows a coupling of the fields $\mathbf{X}^{\mathbf{x}}$ for all boundary conditions $\mathbf{x} \in E^{\mathbb{N}^+} \times E^{\mathbb{N}^+}$ by using the recursion

$$X_{i,j}^{\mathbf{x}} = g(Z_{i,j}, V_{i,j}, U_{i,j}; X_{i-1,j}^{\mathbf{x}}, X_{i,j-1}^{\mathbf{x}}), \quad i, j = 1, 2, \dots \quad (15)$$

along any increasing order of \mathbb{N}^{+2} starting from the boundary conditions

$$X_{i,0}^{\mathbf{x}} = x_{i,0}, X_{0,i}^{\mathbf{x}} = x_{0,i} \quad i = 1, 2, \dots \quad (16)$$

The following definition will be useful in the sequel.

Definition 2.4. For any subset $S \subset \mathbb{Z}^2$ its external boundary, indicated with $\overrightarrow{\partial} S$, is the set of parents of some elements of S , which are not themselves in S . Finally define the internal boundary of S , indicated with $\partial_I S$, as the set of vertices in S with at least one parent in $\overrightarrow{\partial} S$.

Notice that the external boundary of the finite box $\Lambda_{m,n}$ is the set of sites $\{(i, 0), (0, j), i = 1, \dots, m, j = 1, \dots, n\}$.

It is immediately seen that we can use the representation (15) to couple the field in any finite subset S of \mathbb{Z}^2 for all possible boundary conditions prescribed on its external boundary $\overrightarrow{\partial} S$, by recursion along any increasing order of the sites in S . For any choice of boundary conditions $\mathbf{x} = \{x_{i,j} \in E, (i, j) \in \overrightarrow{\partial} S\}$ the unilateral field $\mathbf{X}_S^{\mathbf{x}}$ on S , with b.c.'s \mathbf{x} and transition kernel K , can be therefore represented as

$$\mathbf{X}_S^{\mathbf{x}} = G^S(\mathbf{Z}_S, \mathbf{V}_S, \mathbf{U}_S; \mathbf{x}) \quad (17)$$

where the function G^S is suitably defined.

Furthermore we denote by $G_B^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S; \mathbf{x})$ the projection of the vector-valued function $G^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S; \mathbf{x})$ on the sites belonging to the subset $B \subset S$.

By the direct inspection of (11) we notice that if $\mathbf{Z}_{\partial_I S} = \mathbf{0}$ then the recursion of (15) can be started with the configuration $\mathbf{V}_{\partial_I S}$ irrespectively of the b.c.'s $\mathbf{x} \in E^{\vec{\partial} S}$. We denote by $\Gamma^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S)$ the resulting configuration of the field on S : hence

$$\mathbf{Z}_{\partial_I S} = \mathbf{0} \Rightarrow G^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S; \mathbf{x}) = \Gamma^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S). \quad (18)$$

The next lemma generalizes this situation.

Lemma 2.5. *Let S be a finite subset of \mathbb{Z}^2 , and $B \subset S$. If $\mathbf{Z}_{\partial_I B} = \mathbf{0}$, then*

$$G_B^S(\mathbf{Z}_S, \mathbf{U}_S, \mathbf{V}_S; \mathbf{x}) = \Gamma^B(\mathbf{Z}_B, \mathbf{U}_B, \mathbf{V}_B), \quad \forall \mathbf{x} \in E^{\vec{\partial} S}. \quad (19)$$

Proof. By the argument above the restriction of the field on B does not depend on its values on $\vec{\partial} B$. By consequence it does not depend on the b.c.'s \mathbf{x} on $\vec{\partial} S$. \square

3 Oriented percolation and coupling

In this section the coupling made above is exploited to construct, under Assumption 2.1, a perfect simulation algorithm for the equilibrium distribution of an ergodic unilateral field in any finite box $\Lambda_{m,n}$. Since we keep the box fixed, we write it simply as Λ . Instead, the boundary conditions are pushed far away from Λ , and the field is constructed using a finite number of samples of a given realization of the auxiliary fields \mathbf{Z} , \mathbf{V} and \mathbf{U} . In particular the auxiliary field \mathbf{Z} will play a crucial role, since it indicates when the construction becomes insensitive to the boundary conditions.

We begin by defining an (increasing) *path* γ joining two vertices as a sequence of vertices $\{(i_k, j_k) \in \mathbb{Z}^2\}_{k=0,1,\dots,m}$, with $m > 0$, such that for $k = 0, \dots, m-1$ either

$$i_{k+1} = i_k + 1, j_{k+1} = j_k \quad \text{or} \quad i_{k+1} = i_k, j_{k+1} = j_k + 1. \quad (20)$$

In this case we say that (i_0, j_0) and (i_m, j_m) are joined by the path γ of length m . Obviously two distinct vertices $(a, b) \in \mathbb{Z}^2$ and $(c, d) \in \mathbb{Z}^2$ can be joined by a path if and only if $a \leq c$ and $b \leq d$.

Given a realization of the field $\mathbf{Z} = \{Z_{i,j} \in \{0,1\}, (i,j) \in \mathbb{Z}^2\}$ we say that a path $\gamma = \{(i_k, j_k)\}_{k=0,1,\dots,m}$ is *open* (in the field \mathbf{Z}) if $Z_{i_k, j_k} = 1$ for $k = 1, \dots, m$. By convention we do not require that $Z_{i_0, j_0} = 1$.

We are now interested to study the following random subset of \mathbb{Z}^2

$$\omega(\Lambda) = \omega(\Lambda, \mathbf{Z}) = \{(i, j) \in \mathbb{Z}^2 : \exists \text{ an open path in } \mathbf{Z} \text{ joining } (i, j) \text{ to } (k, l) \in \partial_I \Lambda\} = \bigcup_{(k,l) \in \partial_I \Lambda} C_{k,l} \quad (21)$$

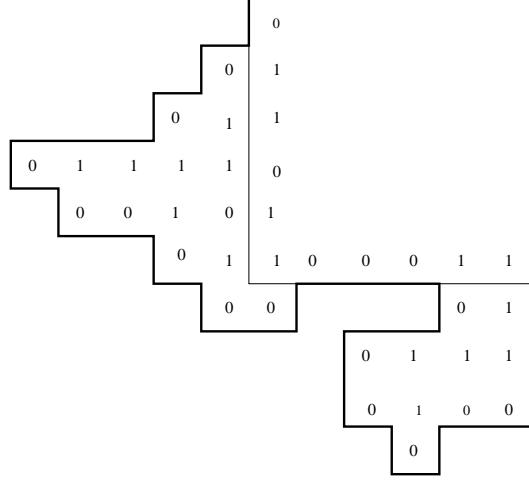


Figure 1: a particular realization of $\omega(\Lambda)$.

where $C_{k,l}$ is the set of vertices joined to $(k,l) \in \partial_I \Lambda$ by an open path in \mathbf{Z} (see Figure 1 for a particular realization).

A fundamental result of oriented percolation for Bernoulli fields (for a general reference on this subject see [1]) ensures that if the probability of 1 at a site does not exceed a critical value $0 < p_c < 1$ then $C_{k,l}$ is finite with probability 1 (irrespectively of (k,l) by translation invariance). Moreover if such probability is strictly smaller than p_c then $\mathbb{E}|C_{k,l}| = \mathbb{E}|C_{0,0}| < \infty$ (see again [1]), which implies that

$$\mathbb{E}|\omega(\Lambda)| \leq |\partial_I \Lambda| \cdot \mathbb{E}|C_{0,0}|. \quad (22)$$

Since an estimate from below is trivially obtained using the bound

$$|\omega(\Lambda)| \geq \sum_{(i,j) \in \partial_I \Lambda} Z_{i,j}, \quad (23)$$

we conclude in this case that $\mathbb{E}|\omega(\Lambda)| = O(|\partial_I \Lambda|)$.

From now on we set $\delta_0 = 1 - p_c$ in Assumption 2.1. Thus

Proposition 3.1. *Under Assumption 2.1 the random subset $\omega(\Lambda)$ is finite almost surely. Moreover if $\delta > \delta_0$ then the mean value of $|\omega(\Lambda)|$ is $O(|\partial_I \Lambda|)$.*

Since we are interested in bounding δ_0 from above, we need to bound p_c from below. For example in [7] it is proved that $p_c \geq 0.682\dots$, thus for $\delta \geq 0.317\dots$ Assumption 2.1 is verified.

Let us define the random set $B(\Lambda) = \Lambda \cup \omega(\Lambda)$. Then we can prove the following

Lemma 3.2. *For any $(i,j) \in \partial_I B(\Lambda)$ it is $Z_{i,j} = 0$.*

Proof. We first notice that the internal boundary $\partial_I B(\Lambda)$ is the disjoint union of $\{(i, j) : (i, j) \in \partial_I \Lambda, Z_{i,j} = 0\}$ and $\partial_I \omega(\Lambda)$. Therefore, to prove the lemma, we need only to show that $Z_{i,j} = 0$ for any $(i, j) \in \partial_I \omega(\Lambda)$. This is due to the fact that if $Z_{i,j} = 1$ then both the parents $(i-1, j)$ and $(i, j-1)$ are in $\omega(\Lambda)$ since the existing open path from (i, j) to a site in $\partial_I \Lambda$ can be extended in both directions. But this is absurd, since $(i, j) \in \partial_I \omega(\Lambda)$. \square

At this point the main result of the paper can be proved. As before the function Γ_A^B denotes the restriction of Γ^B to the sites in $A \subset B$.

Theorem 3.3. *Under Assumption 2.1 the kernel K is uniformly ergodic. Moreover for any finite box $\Lambda = \Lambda_{m,n}$*

$$\Gamma_\Lambda^{B(\Lambda)}(\mathbf{Z}_{B(\Lambda)}, \mathbf{V}_{B(\Lambda)}, \mathbf{U}_{B(\Lambda)}) \sim \mathbf{X}'_\Lambda, \quad (24)$$

where \mathbf{X}' is the equilibrium field with kernel K .

Proof. By Proposition 3.1 $B = B(\Lambda)$ is almost surely finite. Next, for any finite S such that $\Lambda \subset S$, let Q_S^Λ be the event $\{B(\Lambda) \subset S\}$. From Lemma 2.5 and Lemma 3.2 we know that if Q_S^Λ occurs, for any $\mathbf{x} \in E^{\vec{\partial} S}$

$$\mathbf{X}_{S|\Lambda}^\mathbf{x} = G_\Lambda^S(\mathbf{Z}_S, \mathbf{V}_S, \mathbf{U}_S; \mathbf{x}) = \Gamma_\Lambda^B(\mathbf{Z}_B, \mathbf{V}_B, \mathbf{U}_B). \quad (25)$$

Next, let $\{S_k \supset \Lambda, k = 1, 2, \dots\}$ be any sequence of finite subsets increasing to the set of all sites $(h, l) \in \mathbb{Z}^2$ such that $h \leq i$ and $j \leq l$. Then, under Assumption 2.1, the sequence $\{Q_{S_k}^\Lambda\}$ increases to an event of probability 1. This means that with probability 1 $B(\Lambda) \subset S_k$ eventually with k , hence

$$G_\Lambda^{S_k}(\mathbf{Z}_{S_k}, \mathbf{V}_{S_k}, \mathbf{U}_{S_k}; \mathbf{x}_k) = \Gamma_\Lambda^B(\mathbf{Z}_B, \mathbf{V}_B, \mathbf{U}_B) \quad \forall \mathbf{x}_k \in E^{\vec{\partial} S_k} \quad (26)$$

eventually in k .

Next let us consider any sequence $\{(h_k, v_k) k = 1, \dots\}$ converging towards ∞ and construct the sequence of boxes

$$C_k := \{1, \dots, h_k + m\} \times \{1, \dots, v_k + n\} \subset (h_k, v_k) + \Lambda =: \Lambda_k.$$

It is clear that for any b.c. $\mathbf{x}_k \in E^{\vec{\partial} C_k}$

$$\mathbf{X}_{(h_k, v_k) + \Lambda}^{\mathbf{x}_k} = G_{\Lambda_k}^{C_k}(\mathbf{Z}_{C_k}, \mathbf{V}_{C_k}, \mathbf{U}_{C_k}; \mathbf{x}_k) \sim G_\Lambda^{S_k}(\mathbf{Z}_{S_k}, \mathbf{V}_{S_k}, \mathbf{U}_{S_k}; \mathbf{x}_k) \quad (27)$$

where

$$S_k = \{-h_k + 1, \dots, m\} \times \{-v_k + 1, \dots, n\},$$

with the b.c.'s \mathbf{x}_k at the r.h.s. of (27) defined on

$$\vec{\partial} S_k = \{(-h_k, -v_k + 1), \dots, (-h_k, m)\} \cup \{(-h_k + 1, -v_k), \dots, (m, -v_k)\}.$$

The r.h.s of (27) is equal to $\Gamma_\Lambda^B(\mathbf{Z}_B, \mathbf{V}_B, \mathbf{U}_B)$ on $Q_{S_k}^\Lambda$. By consequence, denoting by I_A the indicator of A

$$\|\mathcal{L}(X_{(h_k, v_k) + \Lambda}^{\mathbf{x}_k}) - \mathcal{L}(\Gamma_\Lambda^B)\| = \|\mathcal{L}(G_\Lambda^{S_k}) - \mathcal{L}(\Gamma_\Lambda^B)\| = \sup_A \mathbb{E}(|I_A(G_\Lambda^{S_k}) - I_A(\Gamma_\Lambda^B)|) \leq 2P(Q_{S_k}^\Lambda)^c, \quad (28)$$

from which we deduce uniform convergence in variation because $\lim_{k \rightarrow \infty} P(Q_{S_k}^\Lambda)^c = 0$. Since, by varying Λ , this specifies a stationary and compatible family, by Kolmogorov's Theorem it is a realization of a stationary field \mathbf{X}' in the finite box Λ . \square

To summarize we have constructed a sampling scheme for the equilibrium field \mathbf{X}' on a finite box Λ , based on a random but a.s. finite number of samples from the fields $\mathbf{Z}, \mathbf{V}, \mathbf{U}$. In fact, by Proposition 3.1 this number is proportional to $|\Lambda|$ plus $O(|\partial_I \Lambda|)$ in the average.

We conclude the section by discussing a possible implementation of the algorithm. We construct $\omega(\Lambda)$ by backward induction in the following way. Let

$$\Delta_0 = \{(i, j) : (i, j) \in \partial_I \Lambda, Z_{i,j} = 1\}$$

Then for $k \geq 1$ we determine Δ_{k+1} from Δ_k as

$$\Delta_{k+1} = \{(i, j) : \{(i+1, j), (i, j+1)\} \cap \Delta_k \neq \emptyset, Z_{i,j} = 1\}$$

until the index k_{max} such that $\Delta_{k_{max}+1} = \emptyset$ for the first time, which is finite with probability 1 if $\delta \leq \delta_0$. Then $\omega(\Lambda)$ is the union of $\cup_{k=1}^{k_{max}} \Delta_k$ and its external boundary. Then order the sites in $\omega(\Lambda)$ by starting with those in such an external boundary, then those in $\Delta_{k_{max}}$, next those in $\Delta_{k_{max}-1}$ which were not already in $\Delta_{k_{max}}$, and so on. After having totally ordered all sites in $\omega(\Lambda)$, we start with those in Λ . No matter which order is chosen within each of these regions, the recursion along this total order will allow to compute $\Gamma_\Lambda^B(\mathbf{Z}_B, \mathbf{V}_B, \mathbf{U}_B)$.

4 A class of more general block algorithms

In the previous section we have shown how to construct a perfect simulation scheme for some stationary ergodic unilateral fields on \mathbb{Z}^2 . We have used percolation arguments on a suitable auxiliary Bernoulli field to show that the algorithm works under the "sufficiently large" minorization condition Assumption 2.1. On the other hand, in the one-dimensional case the same idea leads to an algorithm for the exact simulation of a sample from a stationary discrete-time Markov chain that works under any non trivial minorization condition, since in any non degenerate one-dimensional Bernoulli field clusters are always finite.

More generally, the Multigamma coupler of Murdoch and Green [13] extends the above idea to cover the whole class of uniformly ergodic kernels by considering a suitable power of the kernel. In fact, by Theorem 16.0.2 in [14] uniformly ergodic kernels are characterized by a minorization condition on some power K^m of the kernel K , which means that we can

apply the same algorithm to the m -skeleton chain $\{X_{km}, k = 0, 1, \dots\}$. Moreover provided m is large enough the value of δ in the minorization condition can be taken arbitrarily close to 1. More generally, it has been proved in [3] that a vertical backward coupling time exists only for such a class of Markov chains.

Based on these observations, in this section we try to extend the previous results to a wider class of unilateral fields by considering suitable skeleton fields, which in general consist of *blocks* of sites.

By a diagonal of \mathbb{Z}^2 we mean a set

$$D_h = \{(i, j) \in \mathbb{Z}^2 : i + j = h\}$$

for $h \in \mathbb{Z}$. The distance between D_h and D_k is defined as $|h - k|$. The binary field used to indicate the region where the simulation have to be performed is in general defined over a new lattice associated to blocks of l adjacent sites lying on diagonals at distance $(d - 1)l$ one from the other, for given integers $l = 1, 2, \dots$ and $d = 2, 3, \dots$. We start by defining the block

$$B_{0,0} = \{(1, l), (2, l - 1) \dots, (l, 1)\} \quad (29)$$

and, for any pair $(i, j) \in \mathbb{Z}^2$ such that $i + j = 0 \bmod (d - 1)$, define its translates

$$B_{i,j} = B_{0,0} + (il, jl) \quad (30)$$

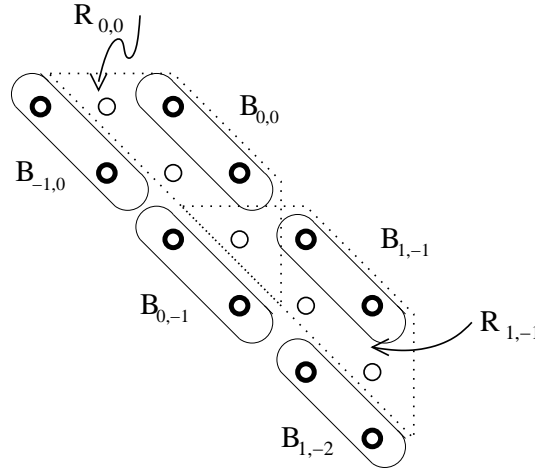


Figure 2: Blocks with $l = 2$ and $d = 2$.

These blocks are taken to be the vertices of a new graph $\mathcal{G}_d = (\mathcal{V}_d, \mathcal{E}_d)$. An oriented edge connects $B_{h,k}$ with $B_{h+i, k+d-1-i}$ for $i = 0, 1, \dots, d - 1$, for any pair $(h, k) \in \mathbb{Z}^2$ such that $h + k = 0 \bmod (d - 1)$. In the following we will directly identify \mathcal{V}_d with such a subset of sites of \mathbb{Z}^2 and refer to the sites $(h + i, k - i - d + 1) \in \mathcal{V}_d$ as the d parents of the site $(h, k) \in \mathcal{V}_d$, for $i = 0, 1, \dots, d - 1$. It is clear that for $l = 1, d = 2$ we get that the graph \mathcal{G}_d is the original lattice \mathbb{Z}^2 ; more generally, for larger values of l and $d = 2$, the graph \mathcal{G}_d

is isomorphic to \mathbb{Z}^2 . In the next section we will devote Example 2 to show that it may be convenient to let l grow, rather than d .

Next let $\mathcal{S} = \{s_m, m = 0, 1, \dots, (d-1)l\}$ be a family of coupling functions, i.e. for $m = 0, 1, \dots, (d-1)l$, $s_m : (0, 1) \times E^2 \rightarrow E$ is separately measurable in each of its arguments and such that

$$s_m(U; x_1, x_2) \sim K(\cdot | x_1, x_2), \quad (31)$$

when U is a uniformly distributed random variable taking values in $(0, 1)$. Since E is Borel the function g defined in (11) is a particular example of (31), since all the auxiliary random variables Z , V and U which are required can be constructed as functions of a single uniformly distributed random variable. The construction of the field $\mathbf{X}_S^\mathbf{x}$ with kernel K over any finite region S with b.c. \mathbf{x} on $E^{\vec{\partial}S}$ is then performed by using a vector \mathbf{U}_S of independent uniformly distributed random variables, iterating along an increasing total order of the sites in S the recursion

$$X_{i,j}^\mathbf{x} = s_{m(i,j)}(U_{i,j}; X_{i-1,j}^\mathbf{x}, X_{i,j-1}^\mathbf{x}) \quad (32)$$

where

$$m(i, j) = i + j - l - 1, \text{ mod}\{(d-1)l\},$$

starting from

$$X_{\vec{\partial}S}^\mathbf{x} = \mathbf{x}.$$

Notice that the coupling function s_m used to construct the value of the field at a given site (i, j) is allowed to depend on $m(i, j)$, the distance of the diagonal where (i, j) lies from the “previous” diagonal of blocks. In Example 1 we will show the usefulness of allowing couplings depending on the diagonal.

Next, for any fixed pair of integers $(h, k) \in \mathcal{V}_d$, consider the trapezoidal region

$$R_{h,k} = \{(i, j) : i \leq l(h+1), j \leq l(k+1), (h+k-d+2)l+2 \leq i+j \leq (h+k+1)l+1\}.$$

The reason for defining this region is that $B_{h,k} \subset R_{h,k}$ and

$$\vec{\partial} R_{h,k} = \cup_{i=0}^{d-1} B_{h-i, k+i-d+1}.$$

Therefore we may represent the field $\mathbf{X}_{R_{h,k}}^\mathbf{x}$ as a function of the vector $\mathbf{U}_{R_{h,k}}$ with i.i.d. components and the boundary values on the parent blocks

$$\mathbf{x} = (\mathbf{x}_{B_{h-i, k+i-d+1}}, i = 0, \dots, d-1)$$

in particular

$$\mathbf{X}_{B_{h,k}}^\mathbf{x} = F^\mathcal{S}(\mathbf{U}_{R_{h,k}}; \mathbf{x}). \quad (33)$$

where $F^\mathcal{S}$ is defined through the recursive application of (32) along an increasing total order of the sites in $R_{h,k}$.

We are now ready to make the following general assumption.

Assumption 4.1. *There exists a binary field*

$$W_{h,k} = \psi(\mathbf{U}_{R_{h,k}}), \quad (h,k) \in \mathcal{V}_d$$

with

$$P(W_{h,k} = 0) = \tilde{\delta} > \frac{d-1}{d}, \quad (34)$$

such that $\{W_{h,k} = 0\}$ implies

$$F^{\mathcal{S}}(\mathbf{U}_{R_{h,k}}; \mathbf{x}) = \Phi(\mathbf{U}_{R_{h,k}}), \quad \forall \mathbf{x} \in E^{\vec{\partial} R_{h,k}} \quad (35)$$

for some measurable function Φ .

We kept Assumption 4.1 quite general in order to accomodate various possible definition of the field $\mathbf{W} = \{W_{h,k}, (h,k) \in \mathcal{V}_d\}$, for a given family \mathcal{S} of coupling functions.

In principle (at least when the state space E is countable) the field \mathbb{W} can be directly defined to have the value zero if and only if $F^{\mathcal{S}}(\mathbf{U}_{R_{h,k}}; \mathbf{x})$ does not depend on $\mathbf{x} \in E^{\vec{\partial} R_{h,k}}$, and in this case $\Phi(\mathbf{U}_{R_{h,k}})$ is equal to such a common value. This means that in order to conclude that $W_{h,k} = 0$ we have to check that the realizations of the field $\mathbf{X}_{B_{h,k}}^{\mathbf{x}}$ started from all the possible b.c.'s $\mathbf{x} \in E^{\vec{\partial} R_{h,k}}$ collapse into a single value. Thus, within a single region $R_{h,k}$ this is similar to the original Propp and Wilson coupling from the past algorithm [16].

A computationally less demanding choice is to define recursively the random subset of E

$$I_{i,j} = s_{m(i,j)}(U_{i,j}; I_{i,j-1} \times I_{i-1,j}), \quad (36)$$

for any site in $(i,j) \in R_{h,k}$, starting from $I_{i,j} = E$ for $(i,j) \in \vec{\partial} R_{h,k}$. Finally we define $W_{h,k} = 0$ if $I_{i,j}$ is a singleton for all $(i,j) \in B_{h,k}$, and $\Phi(\mathbf{U}_{R_{h,k}})$ is then equal to its unique element. Another option for the definition of \mathbf{W} will be presented in Example 1.

A generalization of the coupling (32) is possible by allowing $s_{m(i,j)}(\cdot; X_{i-1,j}^{\mathbf{x}}, X_{i,j-1}^{\mathbf{x}})$ to depend on $U_{l,n}$, $(l,n) \in R_{h,k}$, with $l \leq i$ and $n \leq j$, in such a way that

$$s_{m(i,j)}(U_{l,n}, (l,n) \in R_{h,k}, l \leq i, n \leq j; X_{i-1,j}^{\mathbf{x}}, X_{i,j-1}^{\mathbf{x}}) \sim K(\cdot | X_{i-1,j}^{\mathbf{x}}, X_{i,j-1}^{\mathbf{x}}).$$

A particular example is when $s_{m(i,j)}$ realizes the maximal coupling of the laws $\{K(\cdot | x_1, x_2), (x_1, x_2) \in I_{i-1,j} \times I_{i,j-1}\}$ where $I_{i,j}$ is defined as in (36).

Now consider any finite subset $S \subset \mathbb{Z}^2$. By analogy with Lemma 2.5 it is quite clear that if the set $B \subset S$ is such that $\partial_I B$ is contained in a union of blocks $B_{h,k}$ where $W_{h,k} = 0$, then the restriction of the field $\mathbf{X}_S^{\mathbf{x}}$ on B can be represented as a measurable function $\Upsilon^B(\mathbf{U}_B)$, irrespectively of the b.c.'s $\mathbf{x} \in \vec{\partial} S$.

We say that $\gamma = \{(i_k, j_k) \in \mathcal{V}_d\}_{k=0,1,\dots,m}$ is an increasing path of length m if for $k = 0, \dots, m-1$ the vertex (i_k, j_k) is a parent of (i_{k+1}, j_{k+1}) in the graph \mathcal{G}_d . The field \mathbf{W} is then used to define open paths in \mathcal{G}_d .

Notice that a field $W_{h,k} = \psi(\mathbf{U}_{R_{h,k}})$ defined for $(h,k) \in \mathcal{V}_d$ (see Assumption 4.1) is not a Bernoulli field in general. However since \mathbf{U} is an i.i.d. field the random variables

$$\{W_{h_i,k_i}, (h_i,k_i) \in \mathcal{V}_d, i = 1, \dots, t\}$$

are mutually independent whenever the regions R_{h_i,k_i} are pairwise disjoint, for $i = 1, \dots, t$. Now it can be verified that R_{h_1,k_1} and R_{h_2,k_2} are not disjoint if and only if $h_1 + k_1 = h_2 + k_2 = j(d-1)$ for some integer j and $|h_1 - h_2| \leq d-1$ (e.g. check in Figure 2 that $R_{0,0} \cap R_{1,-1} = \{(2,0)\}$). Therefore the field on each diagonal is $(d-1)$ -dependent and different diagonals are mutually independent.

For the sake of simplicity, in this section we suppose that $\partial_I \Lambda$ is a union of blocks $B_{h,k}$, so it can be identified with a subset of \mathcal{V}_d ; otherwise we can enlarge Λ to have this property. In analogy with (21) we define the random region

$$\sigma(\Lambda) = \sigma(\Lambda, \mathbf{W}) = \{(i,j) \in \mathcal{V}_d : \exists \text{ an open path in } \mathcal{G}_d \text{ joining } (i,j) \text{ to } (k,l) \in \partial_I \Lambda\}. \quad (37)$$

and $D(\Lambda) = \Lambda \cup \sigma(\Lambda)$. As in Lemma 3.2 we can prove that for any $(h,k) \in \partial_I D(\Lambda)$ the random variable $W_{h,k} = 0$. We can finally show the following

Theorem 4.2. *Suppose the kernel K satisfies Assumption 4.1. Then K is uniformly ergodic and for any finite region Λ the law of its stationary version \mathbf{X}'_Λ is the same as $\Upsilon^{D(\Lambda)}(\mathbf{U}_{D(\Lambda)})$. Moreover $\mathbb{E}|D(\Lambda)| = |\Lambda| + O(|\partial_I \Lambda|)$.*

Proof. As in (21)

$$\sigma(\Lambda) = \bigcup_{(h,k) \in \partial_I \Lambda} C'_{h,k}$$

where $C'_{h,k}$ is the set of blocks which can be joined to (h,k) by an open path in \mathcal{G}_d . We have only to prove that under Assumption 4.1 $C'_{0,0}$ is finite almost surely and has finite mean. The only difference with Theorem 3.3 is that \mathbf{W} is not Bernoulli. However, since different diagonals are independent, the values of the field \mathbf{W} on any increasing path are i.i.d. Next define

$$\text{diam}(C'_{0,0}) = \max\{\text{length of an open path from } (h,k) \text{ to } (0,0) : (h,k) \in C'_{0,0}\}. \quad (38)$$

Then the following holds

$$P(|C'_{0,0}| \geq k^2) \leq P(\text{diam}(C'_{0,0}) \geq k) \leq \{d(1 - \bar{\delta})\}^k, \quad (39)$$

where the first inequality is due to a trivial geometric argument. The second inequality follows since the probability that a fixed increasing path of length k is open is equal to $(1 - \bar{\delta})^k$ and the number of such paths leading to the origin is equal to d^k . Since by Assumption 4.1 $d(1 - \bar{\delta}) < 1$ we obtain that $C'_{0,0}$ is finite w.p. 1 and its cardinality has finite mean (in fact, finite moments of any order). \square

It is natural to ask whether it is possible to improve the inequality (34) by dominating \mathbf{W} with a suitable Bernoulli field. A domination result of this type for $(d-1)$ -dependent fields can be found in [11]: however we have checked that it does not improve the bound (34).

Before discussing some examples we need to make clear the limitations of the block algorithm presented in this section, due to the difficulties arising from the two-dimensional structure. In some sense we have constructed a skeleton process $\mathbf{X}_{B_{h,k}}^{\mathbf{x}}$, $(h,k) \in \mathcal{V}_d$: however, this field is not a \mathcal{G}_d -unilateral field anymore, due to the overlap between the regions $R_{h,k}$, for $(h,k) \in \mathcal{V}_d$. As a consequence a minorization condition of the type

$$P(\mathbf{X}_{B_{h,k}}^{\mathbf{x}} \in A | X_{\partial R_{h,k}}^{\mathbf{x}} = \mathbf{x}) \geq \delta \varphi(A)$$

for all $\mathbf{x} \in E^{\vec{\partial} R_{h,k}}$, does not immediately translate in the possibility of coupling the field with an auxiliary Bernoulli field with probability δ of 0. For this reason we need the stronger Assumption 4.1, which requires the choice of a specific family of functions $\{s_m\}$ allowing to simulate the field also at sites common to different regions. Moreover, since the binary auxiliary field used by the algorithm is not Bernoulli, the values of δ for which we can prove that the algorithm works have to be larger than those obtained with i.i.d. percolation ($\delta > 1/2$ rather than the previously cited bound $\delta \geq 0.317\dots$, for $d=2$).

As a consequence we cannot identify the class of uniformly ergodic fields which satisfy Assumption 4.1 for some choice of l and d . As a matter of fact we are not aware of any simple characterizations of the class of uniformly ergodic unilateral random fields, as possible in the one-dimensional case.

5 Examples

Example 1. In this example we will construct a perfect simulation algorithm for the stationary unilateral field with kernel K in a finite region under the following

Assumption 5.1. *There exists a measurable subset $C \subset E$, a probability measure ϕ on E and positive constants $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ with $\rho_1^2 \rho_2 > 2/3$ such that*

$$K(C|y_1, y_2) \geq \rho_1 \quad \forall (y_1, y_2) \in E^2 \quad (40)$$

and

$$K(A|y_1, y_2) \geq \rho_2 \phi(A) \quad \forall (y_1, y_2) \in C^2 \quad (41)$$

for every measurable set A .

Under Assumption 5.1 we will build a coupling of the field with kernel K which satisfies Assumption 4.1 with $l=1$ and $d=3$. In this particular case we need to distinguish between even and odd numbered diagonals, depending on the parity of the sum of the coordinates of a site: the blocks will be single vertices (h,k) lying on even numbered diagonals.

The field will be constructed by means of two families of functions s_0 and s_1 with the property (31), which are used to get the value at sites lying on even and odd numbered diagonals, respectively. For simplicity of notation we use pairs of random variables (U_1, U_2) which are uniformly distributed in $(0, 1)^2$ rather than a single random variable in $(0, 1)$. The functions s_0 and s_1 are defined by means of some functions $f_i : (0, 1) \times E^2 \rightarrow E$, for $i = 0, 1, 2, 3, 4$, measurable in each of the arguments, with the following properties. For $(y_1, y_2) \in E^2$

$$f_0(U_1; y_1, y_2) = f_0(U_1) \sim \phi(\cdot), \quad (42)$$

$$f_1(U_1; y_1, y_2) \sim \begin{cases} \frac{1}{1-\rho_2} [K(\cdot|y_1, y_2) - \rho_2 \phi(\cdot)] & \text{for } y_1, y_2 \in C \\ K(\cdot|y_1, y_2) & \text{otherwise.} \end{cases} \quad (43)$$

$$f_2(U_1; y_1, y_2) \sim K(\cdot|y_1, y_2). \quad (44)$$

$$f_3(U_1; y_1, y_2) \sim \frac{K(\cdot \cap C|y_1, y_2)}{K(C|y_1, y_2)}, \quad (45)$$

$$f_4(U_1; y_1, y_2) \sim \frac{1}{1-\rho_1} \left\{ K(\cdot|y_1, y_2) - \rho_1 \frac{K(\cdot \cap C|y_1, y_2)}{K(C|y_1, y_2)} \right\}. \quad (46)$$

These functions always exist since E is assumed to be Borel. Notice that (46) is well defined since

$$K(\cdot|y_1, y_2) - \rho_1 \frac{K(\cdot \cap C|y_1, y_2)}{K(C|y_1, y_2)} = K(\cdot \cap C^c|y_1, y_2) + \left(1 - \frac{\rho_1}{K(C|y_1, y_2)}\right) K(\cdot \cap C|y_1, y_2) \quad (47)$$

is non-negative by (40).

With these positions it is not difficult to verify that the functions

$$s_0(u_1, u_2; y_1, y_2) = \begin{cases} (1 - 1_{(0, \rho_2)}(u_2))f_0(u_1) + 1_{(0, \rho_2)}(u_2)f_1(u_1; y_1, y_2) & \text{for } y_1, y_2 \in C, \\ f_2(u_1; y_1, y_2) & \text{otherwise.} \end{cases} \quad (48)$$

$$s_1(z, u_1, u_2; y_1, y_2) = 1_{(0, \rho_1)}(u_2)f_3(u_1; y_1, y_2) + (1 - 1_{(0, \rho_1)}(u_2))f_4(u_1; y_1, y_2), \quad (49)$$

satisfy (31). Essentially s_0 and s_1 come from two different mixture decompositions of the kernel. Finally the field \mathbf{W} is defined for any pair $(h, k) \in \mathbb{Z}^2$ such that $(h + k)$ is even by

$$W_{h,k} = 1 - 1_{(0, \rho_1)}((U_2)_{h-1,k})1_{(0, \rho_1)}((U_2)_{h,k-1})1_{(0, \rho_2)}((U_2)_{h,k}). \quad (50)$$

Notice that $W_{h,k} = 0$ if and only if all the indicators appearing in the above formula are equal to 1, which implies that $X_{h,k} = f_0((U_1)_{h,k})$, irrespectively of the values at the parent sites. Since

$$P(W_{i,j} = 0) = \rho_1^2 \rho_2 > \frac{2}{3}$$

we have thus established

Theorem 5.2. *Under Assumption 5.1 the conclusions of Theorem 4.2 hold.*

It is not difficult to construct a kernel K for which Assumption 5.1 is satisfied, whereas Assumption 2.1 is false. Let $E = \{0, 1, 2\}$ and suppose that

$$K(l|y_1, y_2) = \begin{cases} \phi(l) & \text{if } (y_1, y_2) \in \{0, 1\}^2 \\ \delta_{1,l} & \text{if } y_1 = y_2 = 2 \\ \delta_{0,l} & \text{otherwise.} \end{cases} \quad (51)$$

Choosing $C = \{0, 1\}$ (40) holds with $\rho_1 = \phi(C) = \phi(0) + \phi(1)$ and (41) holds with $\rho_2 = 1$, hence if $\phi(C)$ is large enough Assumption 5.1 is satisfied. On the other hand Assumption 2.1 does not hold, since $K(\cdot|2, 2)$ and $K(\cdot|1, 2)$ are singular.

Example 2. This example allows a simple analysis of the role of d and l in the block algorithm. On the state space $E = \{0, 1, 2\}$ consider the unilateral kernel

$$K(l|y_1, y_2) = \begin{cases} p\delta_{2,l} + (1-p)\delta_{1,l} & \text{if } y_1 = y_2 = 2 \\ p\delta_{1,l} + (1-p)\delta_{0,l} & \text{if } \min(y_1, y_2) = 0 \\ p\delta_{\min(y_1, y_2)+1, l} + (1-p)\delta_{\min(y_1, y_2)-1, l} & \text{otherwise.} \end{cases} \quad (52)$$

It is clear that this kernel does not satisfy Assumption 2.1 for any $p \in [0, 1]$, since $K(0|2, 2) = K(1|1, 1) = K(2|0, 0) = 0$. Now assume $p < 1/2$ and consider what happens by increasing d or l . For $d = 3$ and $l = 1$ we see that by choosing $C = \{0, 1\}$ and ϕ the Dirac mass on 0, with $\rho_1 = \rho_2 = 1 - p$, we can fulfil Assumption 5.1 provided $(1 - p)^3 > 2/3$. This choice of C is clearly the best possible.

Next we show that by taking $l = 2$ and $d = 2$ we can enlarge the region of parameters p under which uniform ergodicity can be proved and a perfect simulation algorithm can be constructed through Assumption 4.1. For any site of the lattice we always use the natural coupling $s : (0, 1) \times E^2 \rightarrow E$, defined by

$$s(u; y_1, y_2) = 1_{(0,p)}(u) \min\{2, \min(y_1, y_2) + 1\} + 1_{(p,1)}(u) \max\{0, \min(y_1, y_2) - 1\}. \quad (53)$$

We define $W_{0,0} = 0$ provided $U_{i,j} > p$ on all the sites $(i, j) = (1, 2), (1, 1), (2, 1)$. The whole field $\{W_{h,k}, (h, k) \in \mathbb{Z}^2\}$ is defined by translation. Such a definition ensures that the field has the value zero on both the sites $(2, 1)$ and $(1, 2)$ which form the block $B_{0,0}$, irrespectively of the b.c.'s on the parent blocks, formed by the sites $(-1, 2), (0, 1), (1, 0), (2, -1)$ (see again Figure 2). Hence we can take $\Phi = 0$ in (35). It is immediately obtained that $P(W_{0,0} = 0) = (1 - p)^3$, so that Assumption 4.1 is satisfied provided $(1 - p)^3 > 1/2$, enlarging the region where it is proved that perfect simulation works.

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